(29) Q.No. → If $y = a\cos(\log x) + b\sin(\log x)$, prove that

(i) $x^2 y_2 + x y_1 + y = 0$

(ii) $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$

Ans. → $\because$ $y = a\cos(\log x) + b\sin(\log x)$

D.b.s. w.r.t. x, we have

$$y_1 = a \times -\sin(\log x) \times \frac{1}{x} + b \times \cos(\log x) \times \frac{1}{x}$$

$$y_1 = \frac{1}{x}\left[-a\sin(\log x) + b\cos(\log x)\right]$$

$$xy_1 = -a\sin \log x + b\cos \log x$$

Again D.b.S. w.r.t. x, we have

$y_1 + xy_2 = -a\cos(\log x)\times\frac{1}{x} + b\times -\sin(\log x)\times\frac{1}{x}$

or, $y_1 + xy_2 = -\frac{1}{x}\left[a\cos\log x + b\sin\log x\right]$

or, $xy_1 + x^2y_2 = -y$

$\therefore x^2y_2 + xy_1 + y = 0$ [(i) रहने पर यही साबित हो गया तथा (ii) रहने पर शुरू पूरा लिखना]

Diff. n times by Leibnitz's theorem, we have

$y_{n+2}x^2 + {}^nC_1 y_{n+1}\times 2x + {}^nC_2 y_n \times 2 + \left[y_{n+1}\cdot x + {}^nC_1\cdot y_n\cdot 1\right] + y_n = 0$

or, $y_{n+2}x^2 + 2nxy_{n+1} + \frac{n(n-1)}{2}\times y_n\times 2 + y_{n+1}\cdot x + ny_n + y_n = 0$

or, $y_{n+2}x^2 + xy_{n+1}(2n+1) + \left[n^2 - \cancel{n} + \cancel{n} + 1\right]y_n = 0$

~~or, $y_{n+2}x^2 + xy_{n+1}(2n+1) + (n^2+1)y_n = 0$~~

proved.

or, $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$ proved.

30 (a) QN.→ If $y = \sin(m\sin^{-1}x)$, prove that

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Ans.→ $\because y = \sin(m\sin^{-1}x)$

D.b.S. w.r.t. x, we have

$y_1 = \cos(m\sin^{-1}x)\times\frac{m\times 1}{\sqrt{1-x^2}}$

$y_1\sqrt{1-x^2} = m\cos(m\sin^{-1}x)$

Squaring both sides, we have

$y_1^2(1-x^2) = m^2 \cos^2(m\sin^{-1}x)$

$y_1^2(1-x^2) = m^2[1 - \sin^2(m\sin^{-1}x)]$

$y_1^2(1-x^2) = m^2(1-y^2)$

Again D.b.S. w.r.t. x, we have

$2y_1y_2(1-x^2) + y_1^2 \times -2x = m^2(-2yy_1)$

$2y_1[y_2(1-x^2) - xy_1] = -m^2 2yy_1$

$y_2(1-x^2) - xy_1 = -m^2 y$

① or, $y_2(1-x^2) - xy_1 + m^2 y = 0$ Proved

[(i) रहने पर यहीं साबित है ग
(ii) रहने पर शुरू से पूरा कि
Proved यहाँ नहीं होगा अ
मे ही होगा]

Diff. n times by Leibnitz's theorem, we have

② $y_{n+2}(1-x^2) + {}^nc_1 y_{n+1} \times -2x + {}^nc_2 y_n \times -2 - [y_{n+1}x + {}^nc_1 y_n]$
$+ m^2 y_n = 0$

or, $y_{n+2}(1-x^2) - 2nxy_{n+1} - \frac{2n(n-1)}{2} y_n - xy_{n+1} - ny_n$
$+ m^2 y_n = 0$

or, $y_{n+2}(1-x^2) - xy_{n+1}(2n+1) - y_n(n^2 - n + n - m^2) = 0$

or, $y_{n+2}(1-x^2) - xy_{n+1}(2n+1) - (n^2 - m^2)y_n = 0$

or, $y_{n+2}(1-x^2) - xy_{n+1}(2n+1) + (m^2 - n^2)y_n = 0$ Proved

30 (b) If $y = \cos(m\cos^{-1}x)$ prove that
$(1-x^2)_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

Ans.→ $y = \cos(m\cos^{-1}x)$

D.b.s.w.r.t. x, we have

$$y_1 = -\sin(m\cos^{-1}x) \times \frac{m}{\sqrt{1-x^2}}$$

$$y_1\sqrt{1-x^2} = m\sin(m\cos^{-1}x)$$

Squaring both sides, we have

$$y_1^2(1-x^2) = m^2\sin^2(m\cos^{-1}x)$$

$$y_1^2(1-x^2) = m^2[1-\cos^2(m\cos^{-1}x)]$$

$$y_1^2(1-x^2) = m^2(1-y^2)$$

~~Diff. n times by Leibnitz's theorem, we have~~

Again D.b.s.w.r.t. x, we have

$$2y_1y_2(1-x^2) + y_1^2 \times -2x = m^2(-2yy_1)$$

$$y_2(1-x)^2 - xy_1 = -m^2y$$

$$y_2(1-x^2) - xy_1 + m^2y = 0$$

Diff. n times by Leibnitz's theorem, we have

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1} \times -2x + {}^nC_2\, y_n \times -2 - [y_{n+1}x + {}^nC_1\, y_n] + m^2y_n = 0$$

or, $$y_{n+2}(1-x^2) - 2nxy_{n+1} - \frac{2n(n-1)}{2}y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

or, $y_{n+2}(1-x^2) - xy_{n+1}(2n+1) - y_n(n^2 - \cancel{n} + \cancel{n} - m^2) = 0$

or, $y_{n+2}(1-x^2) - xy_{n+1}(2n+1) - (n^2 - m^2)y_n = 0$

Proved.

(31) QN. → If $y = \sin^{-1}x$, prove that

(i) $(1-x^2)y_2 - xy_1 = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$

Ans. → $\because y = \sin^{-1}x$

D. b. S. w. r. t. x, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

or, $y_1\sqrt{1-x^2} = 1$

~~Again D. b. S. w. r. t. x, we have~~

Squaring both sides, we have

$y_1^2(1-x^2) = 1$

Again D. b. S. w. r. t. x, we have

$2y_1 \cdot y_2(1-x^2) + y_1^2 x - 2x = 1$

$y_2(1-x^2) - xy_1 = 0$ [(i) रहने पर यही साबित हो गया। (ii) रहने पर शुरु से पुरा लिखना]

Diff. n times by Leibnitz's theorem, we have

$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1} x - 2x + {}^nC_2 y_n x - 2 - [y_{n+1} x + {}^nC_1$

$= 0$

$$or,\ y_{n+2}(1-x^2) - 2nx\,y_{n+1} - \frac{n(n-1)}{\cancel{2}} \times y_n \times \cancel{2} - y_{n+1}\cdot x - ny_n = 0$$

$$or,\ y_{n+2}(1-x^2) - x\,y_{n+1}(2n+1) - [n^2 - \cancel{n} + \cancel{n}]\,y_n = 0$$

$$or,\ y_{n+2}(1-x^2) - x\,y_{n+1}(2n+1) - n^2 y_n = 0$$

proved.

(32) If $y = \tan^{-1}x$, prove that ~~$(1+x^2)y_{n+1} + 2nxy_n$~~
$$(1+x)^2 y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$$

Ans → $\because y = \tan^{-1}x$

D.b.S. w.r.t. x, we have,

$$y_1 = \frac{1}{1+x^2}$$

$$(1+x^2)y_1 = 1$$

Diff. n times by Leibnitz's theorem, we have

$$(1+x^2)y_{n+1} + {}^nc_1\, y_n \cdot 2x + {}^nc_2\, y_{n-1}\cdot 2 = 0$$

$$or,\ (1+x^2)y_{n+1} + 2nx\,y_n + \frac{n(n-1)}{\underline{|2}}\cdot y_{n-1} \times 2 = 0$$

$$or,\ (1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$$